Computer Vision

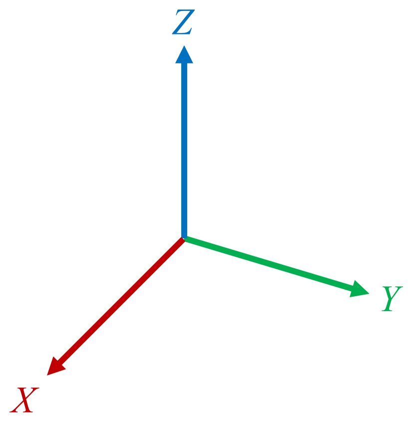
Homogeneous Coordinates

Julien SEINTURIER
Associate Professor

julien.seinturier@univ-tln.fr
http://web.seinturier.fr/teaching/computer-vision

Representing 3D space

- 3D space can be represented with Euclidean Space
  - Vector space on $\mathbb{R}^3$
  - Basis of 3 vectors denoted X, Y and Z
  - dot product $\cdot$ and cross product $\times$



- Mathematical properties
  - Vector orthogonality: $X \cdot Y = X \cdot Z = Y \cdot Z = 0$
  - Plane orthogonality: $X \times Y = Z$, $X \times Z = Y$, $Y \times Z = X$

Representing 3D space

- Limits
  - No uniform representation of transformations
  - Computation can be complex
- Needs
  - A consistent mathematical formalism
  - Good computational properties

Homogeneous Representation

- A vector $\mathcal{X}$ is homogeneous if and only if:

$$\forall \lambda \in \mathbb{R}, \lambda \neq 0, \mathcal{X} = \lambda \mathcal{X}$$

- **Example:**

$$\mathcal{X} = 2\mathcal{X} = 3\mathcal{X} = \lambda \mathcal{X}$$

Homogeneous

$$X \neq 2X \neq 3X \neq \lambda X$$

Euclidean

Homogeneous Representation

- A homogeneous vector $\mathcal{X}$ of dimension n is a tuple made of n homogenous coordinates:

$$\mathcal{X} = (x_1, \dots, x_n)$$

- A homogeneous vector $\mathcal{X}$ of dimension n can also be represented by a column matrix of n rows:

$$\mathcal{X} = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$$

Euclidean to homogeneous Representation

- Let $\mathrm{A} = (a_1, \dots, a_n)$ be a vector within a Euclidean space of dimension n.

  A homogeneous representation of A is a vector of dimension $n+1$, denoted $\mathcal{A}$, such as:

$$\mathcal{A} = \begin{bmatrix} a_1 \\ \cdots \\ a_n \\ 1 \end{bmatrix} = [a_1 \quad \dots \quad a_n \quad 1]^T$$

Euclidean to homogeneous Representation

- **Example:** From Euclidean to homogeneous

$$(1, 4, 3)$$

↑

Euclidean vector

Euclidean to homogeneous Representation

- **Example:** From Euclidean to homogeneous

$$\underset{\text{Euclidean vector}}{(1,4,3)} \rightarrow \underbrace{[1,4,3,1]^T = 2[1,4,3,1]^T = \cdots = \lambda[1,4,3,1]^T}_{\text{Homogeneous representations}}$$

Euclidean to homogeneous Representation

- **Example:** From Euclidean to homogeneous

$$\underset{\uparrow\ \text{Euclidean vector}}{(1, 4, 3)} \rightarrow \underbrace{[1,4,3,1]^T = 2[1,4,3,1]^T = \cdots = \lambda[1,4,3,1]^T}_{\text{Homogeneous representations}}$$

- **Warning:** Homogeneous representation of the Euclidean space origin **is not** the origin of homogeneous space:

$$(0, 0, 0) \rightarrow [0,0,0,1]^T = 2[0,0,0,1]^T = \cdots = \lambda[0,0,0,1]^T$$

Euclidean from homogeneous Representation

- Let $\mathcal{A} = [a_1 \quad \dots \quad a_n \quad w]^T$ be a vector within a homogeneous space of dimension $n+1$, with $w \neq 0$.

  The Euclidean representation of $\mathcal{A}$ is a vector of dimension n, denoted A such as:

$$A = \left(\frac{a_1}{w}, \dots, \frac{a_n}{w}\right)$$

Euclidean from homogeneous Representation

- **Example:** From homogeneous to Euclidean

$$[2 \quad 8 \quad 6 \quad 2]^T$$

homogeneous vector

Euclidean from homogeneous Representation

- **Example:** From homogeneous to Euclidean

$$\underset{\text{homogeneous vector}}{[2 \quad 8 \quad 6 \quad 2]^T} \rightarrow \underbrace{\left(\frac{2}{2},\frac{8}{2},\frac{6}{2}\right) = (1,4,3)}_{\text{Euclidean representation}}$$

Euclidean from homogeneous Representation

- **Example:** From homogeneous to Euclidean

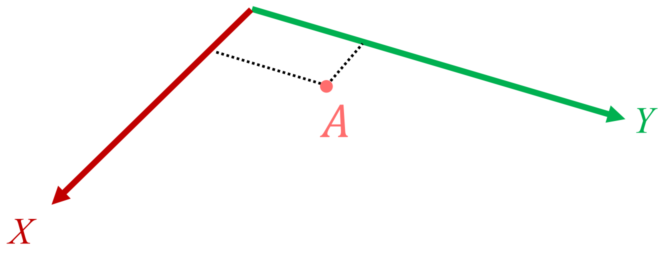
$$\underset{\text{homogeneous vector}}{[2 \quad 8 \quad 6 \quad 2]^T} \rightarrow \underbrace{\left(\frac{2}{2},\frac{8}{2},\frac{6}{2}\right) = (1,4,3)}_{\text{Euclidean representation}}$$

- **Warning:** Homogeneous space origin **cannot be represented** within Euclidean space:

$$[0,0,0,0]^T \rightarrow \left(\frac{0}{0},\frac{0}{0},\frac{0}{0}\right)$$

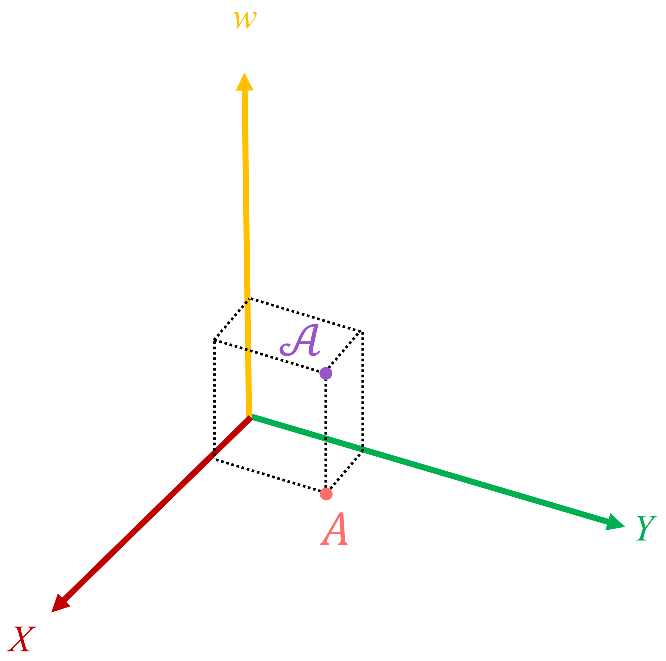
Geometric interpretation

- Let $A = (x, y)$ a vector within a 2D Euclidean space



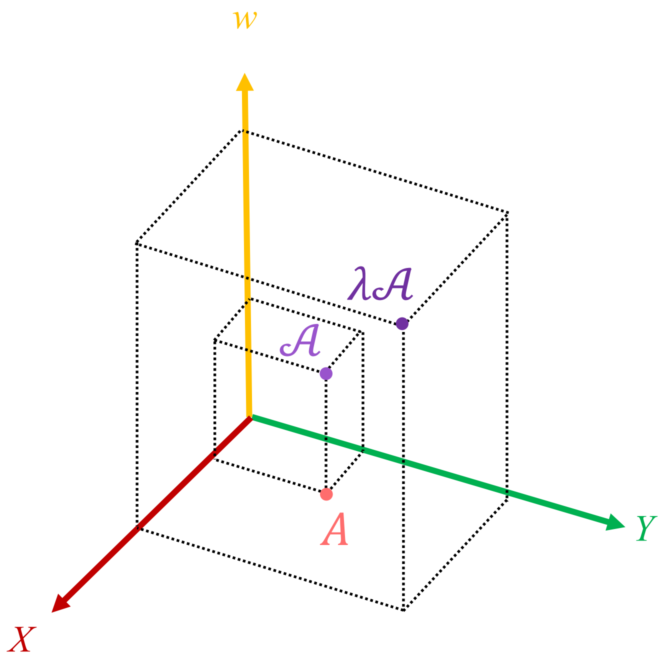
Geometric interpretation

- Let $A = (x, y)$ a vector within a 2D Euclidean space
- $\mathcal{A} = [x \quad y \quad 1]^T$ represents A within the homogeneous space



Geometric interpretation

- Let $A = (x, y)$ a vector within a 2D Euclidean space
- $\mathcal{A} = [x \quad y \quad 1]^T$ represents A within the homogeneous space
- $\lambda\mathcal{A} = [\lambda x \quad \lambda y \quad \lambda]^T$ is another representation of A



Geometric interpretation

- Let $A = (x, y)$ a vector within a 2D Euclidean space
- $\mathcal{A} = [x \quad y \quad 1]^T$ represents A within the homogeneous space
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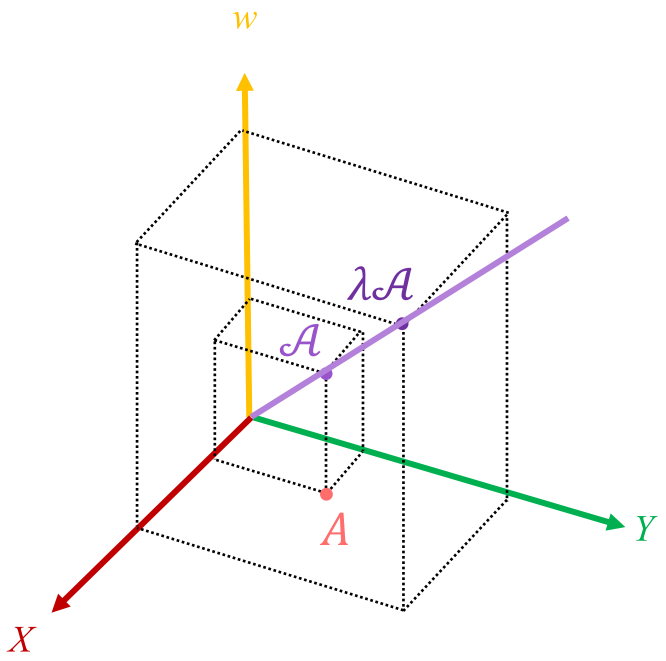

All homogeneous representations of A are lying on the same line

Homogeneous transformation

- Let be a homogeneous space of dimension n and let $\mathcal{V}$ be a homogeneous vector. A homogeneous transform is an application defined such as:

$$\mathcal{H}(\mathcal{V}) = \mathrm{H}\mathcal{V}$$

With:

- H is a $n \times n$ square matrix that represents $\mathcal{H}$
- $\mathrm{H}\mathcal{V}$ is the matrix multiplication of H and $\mathcal{V}$

Homogeneous transformation

- Within 4 dimensioned homogeneous space
  - $\mathcal{V} = [x \quad y \quad z \quad w]^T$
  - H is such as:

$$\mathrm{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

Homogeneous Transformation

- Identity transform

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Computation

$$HV = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity
  - Let $\mathcal{H}$ be a homogeneous transform represented by matrix H:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

  - Let $\mathcal{A}$ and $\mathcal{B}$ two homogeneous vectors represented by column matrices A and B respectively and let λ and μ be two scalars:

$$A = \begin{bmatrix} x_a \\ y_a \\ z_a \\ w_a \end{bmatrix} \quad B = \begin{bmatrix} x_b \\ y_b \\ z_b \\ w_b \end{bmatrix} \quad \lambda A = \begin{bmatrix} \lambda x_a \\ \lambda y_a \\ \lambda z_a \\ \lambda w_a \end{bmatrix} \quad \mu B = \begin{bmatrix} \mu x_b \\ \mu y_b \\ \mu z_b \\ \mu w_b \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \mathrm{H}(\lambda\mathcal{A} + \mu\mathcal{B})$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \mathrm{H}(\lambda A + \mu B)$$

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} \lambda x_a + \mu x_b \\ \lambda y_a + \mu y_b \\ \lambda z_a + \mu z_b \\ \lambda w_a + \mu w_b \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \mathrm{H}(\lambda\mathcal{A} + \mu\mathcal{B})$$

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} \lambda x_a + \mu x_b \\ \lambda y_a + \mu y_b \\ \lambda z_a + \mu z_b \\ \lambda w_a + \mu w_b \end{bmatrix}$$

Matrix product

$$\mathcal{H}(\lambda\mathcal{A} + \mu\mathcal{B}) = \begin{bmatrix} h_{11}(\lambda x_a + \mu x_b) + h_{12}(\lambda y_a + \mu y_b) + h_{13}(\lambda z_a + \mu z_b) + h_{14}(\lambda w_a + \mu w_b) \\ h_{21}(\lambda x_a + \mu x_b) + h_{22}(\lambda y_a + \mu y_b) + h_{23}(\lambda z_a + \mu z_b) + h_{24}(\lambda w_a + \mu w_b) \\ h_{31}(\lambda x_a + \mu x_b) + h_{32}(\lambda y_a + \mu y_b) + h_{33}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \\ h_{41}(\lambda x_a + \mu x_b) + h_{42}(\lambda y_a + \mu y_b) + h_{43}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} \lambda x_a + \mu x_b \\ \lambda y_a + \mu y_b \\ \lambda z_a + \mu z_b \\ \lambda w_a + \mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}(\lambda x_a + \mu x_b) + h_{12}(\lambda y_a + \mu y_b) + h_{13}(\lambda z_a + \mu z_b) + h_{14}(\lambda w_a + \mu w_b) \\ h_{21}(\lambda x_a + \mu x_b) + h_{22}(\lambda y_a + \mu y_b) + h_{23}(\lambda z_a + \mu z_b) + h_{24}(\lambda w_a + \mu w_b) \\ h_{31}(\lambda x_a + \mu x_b) + h_{32}(\lambda y_a + \mu y_b) + h_{33}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \\ h_{41}(\lambda x_a + \mu x_b) + h_{42}(\lambda y_a + \mu y_b) + h_{43}(\lambda z_a + \mu z_b) + h_{44}(\lambda w_a + \mu w_b) \end{bmatrix}$$

Distribution

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{11}\mu x_b + h_{12}\lambda y_a + h_{12}\mu y_b + h_{13}\lambda z_a + h_{13}\mu z_b + h_{14}\lambda w_a + h_{14}\mu w_b \\ h_{21}\lambda x_a + h_{21}\mu x_b + h_{22}\lambda y_a + h_{22}\mu y_b + h_{23}\lambda z_a + h_{23}\mu z_b + h_{24}\lambda w_a + h_{24}\mu w_b \\ h_{31}\lambda x_a + h_{31}\mu x_b + h_{32}\lambda y_a + h_{32}\mu y_b + h_{33}\lambda z_a + h_{33}\mu z_b + h_{44}\lambda w_a + h_{34}\mu w_b \\ h_{41}\lambda x_a + h_{31}\mu x_b + h_{42}\lambda y_a + h_{42}\mu y_b + h_{43}\lambda z_a + h_{43}\mu z_b + h_{44}\lambda w_a + h_{44}\mu w_b \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}(\lambda x_a+\mu x_b)+h_{12}(\lambda y_a+\mu y_b)+h_{13}(\lambda z_a+\mu z_b)+h_{14}(\lambda w_a+\mu w_b) \\ h_{21}(\lambda x_a+\mu x_b)+h_{22}(\lambda y_a+\mu y_b)+h_{23}(\lambda z_a+\mu z_b)+h_{24}(\lambda w_a+\mu w_b) \\ h_{31}(\lambda x_a+\mu x_b)+h_{32}(\lambda y_a+\mu y_b)+h_{33}(\lambda z_a+\mu z_b)+h_{44}(\lambda w_a+\mu w_b) \\ h_{41}(\lambda x_a+\mu x_b)+h_{42}(\lambda y_a+\mu y_b)+h_{43}(\lambda z_a+\mu z_b)+h_{44}(\lambda w_a+\mu w_b) \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a+h_{11}\mu x_b+h_{12}\lambda y_a+h_{12}\mu y_b+h_{13}\lambda z_a+h_{13}\mu z_b+h_{14}\lambda w_a+h_{14}\mu w_b \\ h_{21}\lambda x_a+h_{21}\mu x_b+h_{22}\lambda y_a+h_{22}\mu y_b+h_{23}\lambda z_a+h_{23}\mu z_b+h_{24}\lambda w_a+h_{24}\mu w_b \\ h_{31}\lambda x_a+h_{31}\mu x_b+h_{32}\lambda y_a+h_{32}\mu y_b+h_{33}\lambda z_a+h_{33}\mu z_b+h_{44}\lambda w_a+h_{34}\mu w_b \\ h_{41}\lambda x_a+h_{31}\mu x_b+h_{42}\lambda y_a+h_{42}\mu y_b+h_{43}\lambda z_a+h_{43}\mu z_b+h_{44}\lambda w_a+h_{44}\mu w_b \end{bmatrix}$$

Grouping

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a+h_{12}\lambda y_a+h_{13}\lambda z_a+h_{14}\lambda w_a+h_{11}\mu x_b+h_{12}\mu y_b+h_{13}\mu z_b+h_{14}\mu w_b \\ h_{21}\lambda x_a+h_{22}\lambda y_a+h_{23}\lambda z_a+h_{24}\lambda w_a+h_{21}\mu x_b+h_{22}\mu y_b+h_{23}\mu z_b+h_{24}\mu w_b \\ h_{31}\lambda x_a+h_{32}\lambda y_a+h_{33}\lambda z_a+h_{34}\lambda w_a+h_{31}\mu x_b+h_{32}\mu y_b+h_{33}\mu z_b+h_{34}\mu w_b \\ h_{41}\lambda x_a+h_{42}\lambda y_a+h_{43}\lambda z_a+h_{44}\lambda w_a+h_{41}\mu x_b+h_{42}\mu y_b+h_{43}\mu z_b+h_{44}\mu w_b \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{11}\mu x_b + h_{12}\lambda y_a + h_{12}\mu y_b + h_{13}\lambda z_a + h_{13}\mu z_b + h_{14}\lambda w_a + h_{14}\mu w_b \\ h_{21}\lambda x_a + h_{21}\mu x_b + h_{22}\lambda y_a + h_{22}\mu y_b + h_{23}\lambda z_a + h_{23}\mu z_b + h_{24}\lambda w_a + h_{24}\mu w_b \\ h_{31}\lambda x_a + h_{31}\mu x_b + h_{32}\lambda y_a + h_{32}\mu y_b + h_{33}\lambda z_a + h_{33}\mu z_b + h_{44}\lambda w_a + h_{34}\mu w_b \\ h_{41}\lambda x_a + h_{31}\mu x_b + h_{42}\lambda y_a + h_{42}\mu y_b + h_{43}\lambda z_a + h_{43}\mu z_b + h_{44}\lambda w_a + h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{12}\lambda y_a + h_{13}\lambda z_a + h_{14}\lambda w_a + h_{11}\mu x_b + h_{12}\mu y_b + h_{13}\mu z_b + h_{14}\mu w_b \\ h_{21}\lambda x_a + h_{22}\lambda y_a + h_{23}\lambda z_a + h_{24}\lambda w_a + h_{21}\mu x_b + h_{22}\mu y_b + h_{23}\mu z_b + h_{24}\mu w_b \\ h_{31}\lambda x_a + h_{32}\lambda y_a + h_{33}\lambda z_a + h_{34}\lambda w_a + h_{31}\mu x_b + h_{32}\mu y_b + h_{33}\mu z_b + h_{34}\mu w_b \\ h_{41}\lambda x_a + h_{42}\lambda y_a + h_{43}\lambda z_a + h_{44}\lambda w_a + h_{41}\mu x_b + h_{42}\mu y_b + h_{43}\mu z_b + h_{44}\mu w_b \end{bmatrix}$$

Decomposition

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{12}\lambda y_a + h_{13}\lambda z_a + h_{14}\lambda w_a \\ h_{21}\lambda x_a + h_{22}\lambda y_a + h_{23}\lambda z_a + h_{24}\lambda w_a \\ h_{31}\lambda x_a + h_{32}\lambda y_a + h_{33}\lambda z_a + h_{34}\lambda w_a \\ h_{41}\lambda x_a + h_{42}\lambda y_a + h_{43}\lambda z_a + h_{44}\lambda w_a \end{bmatrix} + \begin{bmatrix} h_{11}\mu x_b + h_{12}\mu y_b + h_{13}\mu z_b + h_{14}\mu w_b \\ h_{21}\mu x_b + h_{22}\mu y_b + h_{23}\mu z_b + h_{24}\mu w_b \\ h_{31}\mu x_b + h_{32}\mu y_b + h_{33}\mu z_b + h_{34}\mu w_b \\ h_{41}\mu x_b + h_{42}\mu y_b + h_{43}\mu z_b + h_{44}\mu w_b \end{bmatrix}$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\begin{bmatrix} h_{11}\lambda x_a+h_{12}\lambda y_a+h_{13}\lambda z_a+h_{14}\lambda w_a+h_{11}\mu x_b+h_{12}\mu y_b+h_{13}\mu z_b+h_{14}\mu w_b \\ h_{21}\lambda x_a+h_{22}\lambda y_a+h_{23}\lambda z_a+h_{24}\lambda w_a+h_{21}\mu x_b+h_{22}\mu y_b+h_{23}\mu z_b+h_{24}\mu w_b \\ h_{31}\lambda x_a+h_{32}\lambda y_a+h_{33}\lambda z_a+h_{34}\lambda w_a+h_{31}\mu x_b+h_{32}\mu y_b+h_{33}\mu z_b+h_{34}\mu w_b \\ h_{41}\lambda x_a+h_{42}\lambda y_a+h_{43}\lambda z_a+h_{44}\lambda w_a+h_{41}\mu x_b+h_{42}\mu y_b+h_{43}\mu z_b+h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\begin{bmatrix} h_{11}\lambda x_a+h_{12}\lambda y_a+h_{13}\lambda z_a+h_{14}\lambda w_a \\ h_{21}\lambda x_a+h_{22}\lambda y_a+h_{23}\lambda z_a+h_{24}\lambda w_a \\ h_{31}\lambda x_a+h_{32}\lambda y_a+h_{33}\lambda z_a+h_{34}\lambda w_a \\ h_{41}\lambda x_a+h_{42}\lambda y_a+h_{43}\lambda z_a+h_{44}\lambda w_a \end{bmatrix}+\begin{bmatrix} h_{11}\mu x_b+h_{12}\mu y_b+h_{13}\mu z_b+h_{14}\mu w_b \\ h_{21}\mu x_b+h_{22}\mu y_b+h_{23}\mu z_b+h_{24}\mu w_b \\ h_{31}\mu x_b+h_{32}\mu y_b+h_{33}\mu z_b+h_{34}\mu w_b \\ h_{41}\mu x_b+h_{42}\mu y_b+h_{43}\mu z_b+h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\lambda\begin{bmatrix} h_{11}x_a+h_{12}y_a+h_{13}z_a+h_{14}w_a \\ h_{21}x_a+h_{22}y_a+h_{23}z_a+h_{24}w_a \\ h_{31}x_a+h_{32}y_a+h_{33}z_a+h_{34}w_a \\ h_{41}x_a+h_{42}y_a+h_{43}z_a+h_{44}w_a \end{bmatrix}+\mu\begin{bmatrix} h_{11}x_b+h_{12}y_b+h_{13}z_b+h_{14}w_b \\ h_{21}x_b+h_{22}y_b+h_{23}z_b+h_{24}w_b \\ h_{31}x_b+h_{32}y_b+h_{33}z_b+h_{34}w_b \\ h_{41}x_b+h_{42}y_b+h_{43}z_b+h_{44}w_b \end{bmatrix}$$

Factorization

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \begin{bmatrix} h_{11}\lambda x_a + h_{12}\lambda y_a + h_{13}\lambda z_a + h_{14}\lambda w_a \\ h_{21}\lambda x_a + h_{22}\lambda y_a + h_{23}\lambda z_a + h_{24}\lambda w_a \\ h_{31}\lambda x_a + h_{32}\lambda y_a + h_{33}\lambda z_a + h_{34}\lambda w_a \\ h_{41}\lambda x_a + h_{42}\lambda y_a + h_{43}\lambda z_a + h_{44}\lambda w_a \end{bmatrix} + \begin{bmatrix} h_{11}\mu x_b + h_{12}\mu y_b + h_{13}\mu z_b + h_{14}\mu w_b \\ h_{21}\mu x_b + h_{22}\mu y_b + h_{23}\mu z_b + h_{24}\mu w_b \\ h_{31}\mu x_b + h_{32}\mu y_b + h_{33}\mu z_b + h_{34}\mu w_b \\ h_{41}\mu x_b + h_{42}\mu y_b + h_{43}\mu z_b + h_{44}\mu w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda\begin{bmatrix} h_{11}x_a + h_{12}y_a + h_{13}z_a + h_{14}w_a \\ h_{21}x_a + h_{22}y_a + h_{23}z_a + h_{24}w_a \\ h_{31}x_a + h_{32}y_a + h_{33}z_a + h_{34}w_a \\ h_{41}x_a + h_{42}y_a + h_{43}z_a + h_{44}w_a \end{bmatrix} + \mu\begin{bmatrix} h_{11}x_b + h_{12}y_b + h_{13}z_b + h_{14}w_b \\ h_{21}x_b + h_{22}y_b + h_{23}z_b + h_{24}w_b \\ h_{31}x_b + h_{32}y_b + h_{33}z_b + h_{34}w_b \\ h_{41}x_b + h_{42}y_b + h_{43}z_b + h_{44}w_b \end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B}) = \lambda \mathrm{H}\mathcal{A} + \mu\mathrm{H}\mathcal{B}$$

Matrix product

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\lambda\begin{bmatrix}h_{11}x_a+h_{12}y_a+h_{13}z_a+h_{14}w_a\\h_{21}x_a+h_{22}y_a+h_{23}z_a+h_{24}w_a\\h_{31}x_a+h_{32}y_a+h_{33}z_a+h_{34}w_a\\h_{41}x_a+h_{42}y_a+h_{43}z_a+h_{44}w_a\end{bmatrix}+\mu\begin{bmatrix}h_{11}x_b+h_{12}y_b+h_{13}z_b+h_{14}w_b\\h_{21}x_b+h_{22}y_b+h_{23}z_b+h_{24}w_b\\h_{31}x_b+h_{32}y_b+h_{33}z_b+h_{34}w_b\\h_{41}x_b+h_{42}y_b+h_{43}z_b+h_{44}w_b\end{bmatrix}$$

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\lambda\mathrm{H}\mathcal{A}+\mu\mathrm{H}\mathcal{B}$$

Definition

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\lambda\mathcal{H}(\mathcal{A})+\mu\mathcal{H}(\mathcal{B})$$

Homogeneous transformation

- Homogeneous transformation linearity

$$\mathcal{H}(\lambda\mathcal{A}+\mu\mathcal{B})=\lambda\begin{bmatrix}h_{11}x_a+h_{12}y_a+h_{13}z_a+h_{14}w_a\\h_{21}x_a+h_{22}y_a+h_{23}z_a+h_{24}w_a\\h_{31}x_a+h_{32}y_a+h_{33}z_a+h_{34}w_a\\h_{41}x_a+h_{42}y_a+h_{43}z_a+h_{44}w_a\end{bmatrix}+\mu\begin{bmatrix}h_{11}x_b+h_{12}y_b+h_{13}z_b+h_{14}w_b\\h_{21}x_b+h_{22}y_b+h_{23}z_b+h_{24}w_b\\h_{31}x_b+h_{32}y_b+h_{33}z_b+h_{34}w_b\\h_{41}x_b+h_{42}y_b+h_{43}z_b+h_{44}w_b\end{bmatrix}$$

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Any homogeneous transformation is a **linear application**

Homogeneous Transformation

- Which kind of transformation can be represented ?
- Is it possible to use Homogeneous transformations to represents Euclidean transformations ?
- Is there transformation that can only be computed using Homogeneous coordinates ?

Homogeneous Translation

- Let $\mathcal{X} = [x, y, z, w]^T$ be a homogeneous vector. The translation of $\mathcal{X}$ along a vector $[\alpha, \beta, \gamma, 1]^T$, denoted $\mathcal{T}(\alpha, \beta, \gamma)$, is such as:

$$\mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X}$$

Where:

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathcal{T}(\alpha,\beta,\gamma)(\mathcal{X}) = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha \\ y+\beta \\ z+\gamma \\ 1 \end{bmatrix}$$

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- Euclidean representation: $[x+\alpha, y+\beta, z+\gamma, 1]^T \rightarrow \left(\frac{x+\alpha}{1}, \frac{y+\beta}{1}, \frac{z+\gamma}{1}\right) = (x+\alpha, y+\beta, z+\gamma)$

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$$\boldsymbol{\mathcal{T}(\alpha,\beta,\gamma)(\mathcal{X})} \rightarrow \boldsymbol{T(\alpha,\beta,\gamma)(X)}$$

Homogeneous translation inverse

- Let $\mathcal{X}$ and $\mathcal{Y}$ be a homogeneous vectors where $\mathcal{Y}$ is the result of the homogeneous translation of $\mathcal{X}$ along a vector $[\alpha, \beta, \gamma, 1]^T$

$$\mathcal{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathcal{Y} = \mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X}) = \mathrm{T}\mathcal{X} = \begin{bmatrix} x + \alpha w \\ y + \beta w \\ z + \gamma w \\ w \end{bmatrix}$$

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$$\mathcal{T}(-\alpha, -\beta, -\gamma)(\mathcal{Y}) = T'\mathcal{Y} = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x + \alpha \\ y + \beta \\ z + \gamma \\ w \end{bmatrix}$$

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$$\mathcal{T}(-\alpha, -\beta, -\gamma)\big(\mathcal{T}(\alpha, \beta, \gamma)(\mathcal{X})\big) = T'T\mathcal{X} = \mathcal{X}$$

A Homogeneous translation $\mathcal{T}(\alpha, \beta, \gamma)$ is invertible and $\mathcal{T}(-\alpha, -\beta, -\gamma)$ is its inverse

Homogeneous translation inverse

- Matrix inverse method

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T' = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous translation inverse

- Matrix inverse method

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Homogeneous translation inverse

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$$\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{T}' = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$TT' = ID$$

T' is the inverse matrix of T

Homogeneous translation

- Definition

$$\mathcal{T}(\alpha,\beta,\gamma) = \mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Properties
  - Linear application
  - Invertible: $\mathcal{T}^{-1}(\alpha,\beta,\gamma) = T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & -\beta \\ 0 & 0 & 1 & -\gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^T$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle ω around X axis, denoted $\mathcal{R}_x(\omega)$, is such as:

$$\mathcal{R}_x(\omega)(\mathcal{V}) = \mathrm{R}_x\mathcal{V}$$

Where:

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathcal{R}_x(\omega)(\mathcal{V}) = \begin{bmatrix} x \\ y\cos\omega - z\sin\omega \\ y\sin\omega + z\cos\omega \\ w \end{bmatrix}$$

Computing Euclidean axis rotation

- Let $V = (x, y, z)$ be a vector within 3D Euclidean space.

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$$\rightarrow (x, y\cos\omega - z\sin\omega, y\sin\omega + z\cos\omega)$$

$$\mathcal{R}_x(\omega)(\mathcal{V}) \rightarrow R_x(\omega)(V)$$

Homogeneous rotation inverse

- Let R_x be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let R'_x be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}'_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\omega) & -\sin(-\omega) & 0 \\ 0 & \sin(-\omega) & \cos(-\omega) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\cos(-\omega) = \cos\omega$$

$$\mathrm{R}'_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin(-\omega) & 0 \\ 0 & \sin(-\omega) & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\sin(-\omega) = -\sin\omega$$

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$$\mathrm{R}_x\mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathrm{R}_x\mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\omega + \sin^2\omega & \cos\omega\sin\omega - \cos\omega\sin\omega & 0 \\ 0 & \sin\omega\cos\omega - \cos\omega\sin\omega & \sin^2\omega + \cos^2\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous rotation inverse

- Let R_x be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let R_x^T be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\sin^2\omega + \cos^2\omega = 1$$

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Homogeneous rotation inverse

- Let R_x be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let R_x^T be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_x\mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \cos\omega\sin\omega - \cos\omega\sin\omega & 0 \\ 0 & \sin\omega\cos\omega - \cos\omega\sin\omega & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathrm{R}_x\mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = ID$$

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- Let R_x be the matrix of the homogeneous rotation $\mathcal{R}_x(\omega)$ and let R_x^T be the matrix of the homogeneous rotation $\mathcal{R}_x(-\omega)$

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$$\mathrm{R}_x\mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = ID$$

A Homogeneous X axis rotation $\mathcal{R}_x(\omega)$ is invertible and $\mathcal{R}_x(-\omega)$ is its inverse. Moreover, $\mathrm{R}_x^{-1} = \mathrm{R}_x^T$

Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^T$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle φ around Y axis, denoted $\mathcal{R}_y(\varphi)$, is such as:

$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \mathrm{R}_y\mathcal{V}$$

Where:

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^T$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle φ around Y axis, denoted $\mathcal{R}_y(\varphi)$, is such as:

$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \mathrm{R}_y\mathcal{V} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

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$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \begin{bmatrix} x\cos\varphi + z\sin\varphi \\ y \\ z\cos\varphi - x\sin\varphi \\ w \end{bmatrix}$$

Computing Euclidean axis rotation

- Let $V = (x, y, z)$ be a vector within 3D Euclidean space.

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$$\mathcal{R}_y(\varphi)(\mathcal{V}) = \begin{bmatrix} x\cos\varphi + z\sin\varphi \\ y \\ z\cos\varphi - x\sin\varphi \\ 1 \end{bmatrix} \to \left(\frac{x\cos\varphi + z\sin\varphi}{1}, \frac{y}{1}, \frac{z\cos\varphi - x\sin\varphi}{1}\right)$$

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$$\rightarrow (x\cos\varphi + z\sin\varphi\,, y, z\cos\varphi - x\sin\varphi)$$

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$$\rightarrow (x\cos\varphi + z\sin\varphi, y, z\cos\varphi - x\sin\varphi)$$

$$\mathcal{R}_y(\varphi)(\mathcal{V}) \rightarrow R_y(\varphi)(V)$$

Homogeneous rotation inverse

- Let R_y be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let R'_y be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}'_y = \begin{bmatrix} \cos(-\varphi) & 0 & \sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\cos(-\omega) = \cos\omega$$

$$\mathrm{R}'_y = \begin{bmatrix} \cos\varphi & 0 & \sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\varphi) & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous rotation inverse

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$$\sin(-\omega) = -\sin\omega$$

$$\mathrm{R}'_x = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous rotation inverse

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Homogeneous rotation inverse

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$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & \cos^2\varphi + \sin^2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous rotation inverse

- Let R_y be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let R_y^T be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^T = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & \cos^2\varphi + \sin^2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sin^2\omega + \cos^2\omega = 1$$

$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} 1 & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous rotation inverse

- Let R_y be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let R_y^T be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^T = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} 1 & 0 & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = ID$$

Homogeneous rotation inverse

- Let R_y be the matrix of the homogeneous rotation $\mathcal{R}_y(\varphi)$ and let R_y^T be the matrix of the homogeneous rotation $\mathcal{R}_y(-\varphi)$

$$\mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_y^T = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{R}_y\mathrm{R}_y^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = ID$$

A Homogeneous Y axis rotation $\mathcal{R}_y(\varphi)$ is invertible and $\mathcal{R}_y(-\varphi)$ is its inverse. Moreover, $\mathrm{R}_y^{-1} = \mathrm{R}_y^T$

Homogeneous axis rotation

- Let $\mathcal{V} = [x, y, z, w]^T$ be a homogeneous vector. The rotation of $\mathcal{V}$ by an angle κ around Z axis, denoted $\mathcal{R}_z(\kappa)$, is such as:

$$\mathcal{R}_z(\kappa)(\mathcal{V}) = \mathrm{R}_z\mathcal{V}$$

Where:

$$\mathrm{R}_z = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous axis rotation

- Transform Matrices

$$\mathrm{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{R}_z = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 & 0 \\ \sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse transform matrices

$$\mathrm{R}_x^{-1} = \mathrm{R}_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{R}_y^{-1} = \mathrm{R}_y^T = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{R}_z^{-1} = \mathrm{R}_z^T = \begin{bmatrix} \cos\varphi & \sin\kappa & 0 & 0 \\ -\sin\kappa & \cos\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$